

LAPLACE OPERATORS WITH EIGENFUNCTIONS WHOSE NODAL SET IS A KNOT

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ABSTRACT. We prove that, given any knot γ in a compact 3-manifold M , there exists a Riemannian metric on M such that there is a complex-valued eigenfunction u of the Laplacian, corresponding to the first nontrivial eigenvalue, whose nodal set $u^{-1}(0)$ has a connected component given by γ . Higher dimensional analogs of this result will also be considered.

1. INTRODUCTION

Let M be a closed manifold of dimension 3 endowed with a smooth Riemannian metric g . The eigenfunctions of M satisfy the equation

$$\Delta u_k = -\lambda_k u_k,$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of M and Δ is the Laplace operator of the manifold. As is well known, the zero set $u_k^{-1}(0)$ is called the *nodal set* of the eigenfunction.

Our goal in this paper is to establish the existence of knotted nodal sets in eigenfunctions of the Laplacian on a Riemannian 3-manifold. Specifically, the starting point of the present paper is the following

Question. Let γ be a (possibly knotted) curve in M . Is there a complex-valued eigenfunction u of the Laplacian whose nodal set $u^{-1}(0)$ is diffeomorphic to γ ?

Before getting into the background of this question, it is worth discussing why the eigenfunction u is assumed complex-valued. The idea is that, by the maximum principle, the nodal set of a real-valued eigenfunction (i.e., $u_k^{-1}(0)$) cannot have a connected component of codimension greater than or equal to 2, so there cannot be a real eigenfunction of the Laplacian whose nodal set has a component diffeomorphic to the curve γ . A way to bypass this obstruction is to take a complex function of the form, say, $u := u_1 + iu_2$, which will indeed be an eigenfunction provided that $\lambda_1 = \lambda_2$. Notice that $u^{-1}(0) = u_1^{-1}(0) \cap u_2^{-1}(0)$. Hence the above question is in fact a problem about Laplacians with degenerate eigenvalues.

The context of this question is the study of the nodal sets of the eigenfunctions of the Laplacian in a compact Riemannian manifold, which is a classical topic in geometric analysis with a number of important open problems [20, 21]. In a way, this question addresses the flexibility of low-energy Laplace eigenfunctions, as we shall explain next.

It is well known that high-energy eigenvalues and eigenfunctions of the Laplacian are rather rigid, as illustrated by Weyl's asymptotic formula for the eigenvalues and by Yau's conjecture concerning the Hausdorff measure of the eigenfunction's nodal

sets as the eigenvalue tends to infinity, which was proved for real-analytic metrics in [6]. Indeed, the measure-theoretic properties of the nodal set of high-energy eigenfunctions have been a hot topic of research for decades [6, 13, 17]; a recent account of the subject can be found in [15].

On the contrary, the situation for low-energy eigenvalues and eigenfunctions is much more flexible. The eigenvalue case was considered by Colin de Verdière in [5], where it was shown that there are metrics whose first N eigenvalues can be prescribed a priori. Concerning nodal sets, the problem of constructing a metric on M for which the nodal set $u_1^{-1}(0)$ of the first eigenfunction has a prescribed connected component was solved in [10] (see also [11, 16] for the two dimensional case).

The question that we have started with addresses the possibility of prescribing a component of the intersection of nodal sets

$$u^{-1}(0) = u_1^{-1}(0) \cap u_2^{-1}(0),$$

subject to the additional constraint that the eigenvalues λ_1 and λ_2 should coincide. This constraint is important, since it is well known that for a generic metric the eigenvalues are all simple [19]. In a way, this problem is the metric counterpart of Berry's conjecture [3], recently proved in [7], which states that there should be a smooth (real) potential for which the Schrödinger operator $-\Delta + V$ in \mathbb{R}^3 admits a complex-valued eigenfunction whose nodal set has a connected component of any given knot type.

Our main result in this paper provides an affirmative answer to the above question. Specifically, we show that given a smooth closed curve γ (without self-intersections, but possibly knotted), there is a metric for which $u^{-1}(0)$ has a connected component given by γ , with $u := u_1 + iu_2$ corresponding to the first eigenvalue $\lambda_1 = \lambda_2$, which has multiplicity 2. Moreover, the knot γ is *structurally stable* in the sense that any small enough perturbation of u (in the C^k norm with $k \geq 1$) has a connected component in its nodal set that is a small perturbation of γ (i.e., the image of γ under a smooth diffeomorphism of M that is close to the identity in the C^k norm). We are thus led to the following:

Theorem 1.1. *Let γ be a (possibly knotted) closed curve in a closed 3-manifold M . Then there exists a Riemannian metric g on M such that its first nontrivial eigenvalue has multiplicity 2 (i.e., $\lambda_1 = \lambda_2 \neq \lambda_3$) and γ is a connected component of the nodal set $u^{-1}(0)$ of the complex-valued eigenfunction $u = u_1 + iu_2$. Moreover, γ is a structurally stable component of the nodal set.*

Equivalently, the theorem asserts that the closed curve γ is a connected component of the intersection of the nodal sets of the real-valued eigenfunctions $u_1^{-1}(0) \cap u_2^{-1}(0)$ for any basis of the eigenspace corresponding to the first nontrivial eigenvalue.

The proof of this result starts off with the construction of a metric g that is of order ϵ in the complement of a tubular neighborhood Ω of the curve γ . If ϵ is small enough, the eigenfunctions u_1 and u_2 are close in the set Ω to the first nontrivial Neumann eigenfunctions v_1 and v_2 of this set, so in principle the nodal set of the former can be controlled in terms of the latter using Thom's isotopy theorem. To develop this strategy there are two difficulties one has to deal with. First, the intersection $v_1^{-1}(0) \cap v_2^{-1}(0)$ must be transverse in order to apply Thom's isotopy

theorem. Second, the eigenvalue λ_1 must be degenerate, which is a non-generic property according to Uhlenbeck's theorem [19]. These problems are overcome by constructing very carefully the metric g in successive steps.

Since the nodal set of the complex-valued eigenfunction $u = u_1 + iu_2$ is the intersection of the nodal sets $u_1^{-1}(0) \cap u_2^{-1}(0)$, the question that we have started our paper with is simply the easiest case of whether we can realize a set of codimension $m \geq 2$ in a compact d -manifold as the intersection of the nodal sets

$$u_1^{-1}(0) \cap \cdots \cap u_m^{-1}(0)$$

of m real-valued eigenfunctions having the same energy $\lambda_1 = \cdots = \lambda_m$ (of multiplicity m). The full problem will be tackled in Section 5, where we provide a higher-dimensional version of Theorem 1.1 (Theorem 5.1). This is a significant generalization of [10], but its statement is a little more technical than that of Theorem 1.1.

The content of this paper is as follows. The proof of Theorem 1.1 is presented in Section 2, although the proofs of two technical results, Propositions 2.1 and 2.2, are relegated to Sections 3 and 4, respectively. Finally, in Section 5 we state and prove a higher dimensional analog of Theorem 1.1.

2. PROOF OF THE MAIN THEOREM

The proof of Theorem 1.1 is divided in four steps. In Step 1 we introduce an appropriate metric g_0 whose explicit expression in a tubular neighborhood Ω of the curve γ allows us to compute, in closed form, the first Neumann eigenvalues and eigenfunctions of the domain Ω . In Step 2 we rescale the metric g_0 with a smooth factor that is of order ϵ in the complement of Ω , and show that the eigenvalues and eigenfunctions of M with this metric tend to the Neumann eigenvalues and eigenfunctions of Ω as $\epsilon \searrow 0$ (Proposition 2.1). In Step 3 we prove that the metric introduced in the previous step can be deformed so that its first nontrivial eigenvalue has multiplicity 2 and its corresponding eigenspace is close to an eigenspace of Neumann eigenfunctions of Ω (Proposition 2.2). Finally, in Step 4 we use Thom's isotopy theorem to control the intersection of the nodal sets of the eigenfunctions obtained before.

Step 1: Neumann eigenfunctions of a certain bounded domain. Let us consider a tubular neighborhood Ω of the curve γ , which is diffeomorphic to the product $\mathbb{S}^1 \times \mathbb{D}^2$ because the normal bundle of a knot is always trivial [18]. Hence we will parametrize the set Ω using coordinates $z = (z_1, z_2) \in \mathbb{D}^2$ and $s \in \mathbb{R}/\mathbb{Z}$. Here \mathbb{D}^2 denotes the unit two-dimensional disk.

We shall take a metric g_0 on M such that its restriction to Ω can be written in these coordinates as

$$g_0|_{\Omega} := A^{-1} ds^2 + |dz|^2 = A^{-1} ds^2 + dr^2 + r^2 d\theta^2.$$

Here (r, θ) are the polar coordinates in the disk associated to z and A is a large enough constant. Specifically, we assume that

$$(2.1) \quad A > \frac{\mu_{\mathbb{D}^2}}{4\pi^2},$$

where $\mu_{\mathbb{D}^2}$ is the first nontrivial Neumann eigenvalue of the unit 2-disk. Notice that the coordinate s essentially corresponds to a parametrization of the curve γ

proportional to arc-length and that the variables z describe a disk orthogonal to the curve.

Let us consider the Neumann eigenvalue problem

$$\Delta v_k = -\mu_k v_k \quad \text{in } \Omega, \quad \partial_\nu v_k = 0 \quad \text{on } \partial\Omega,$$

where Δ stands for the Laplacian associated with the metric g_0 . By separation of variables and using the condition (2.1), we infer that the Neumann eigenvalues μ_k of the domain Ω with respect to the metric g_0 are then $\mu_0 = 0$,

$$\mu_1 = \mu_2 = \mu_{\mathbb{D}^2},$$

and $\mu_k > \mu_{\mathbb{D}^2}$ for $k \geq 3$. Hence the eigenspace of μ_1 is two-dimensional, with a particular orthogonal basis being

$$v_1 = J_1(\sqrt{\mu_1}r) \cos \theta, \quad v_2 = J_1(\sqrt{\mu_1}r) \sin \theta.$$

Here J_1 denotes the Bessel function.

Since $J_1(\sqrt{\mu_1}r)$ does not vanish for $0 < r < 1$ and the eigenfunctions behave in a neighborhood of the origin as

$$v_1 = \frac{\sqrt{\mu_1}}{2}r \cos \theta + O(r^2) = \frac{\sqrt{\mu_1}}{2}z_1 + O(|z|^2), \quad v_2 = \frac{\sqrt{\mu_1}}{2}z_2 + O(|z|^2),$$

it is elementary that, as the eigenfunctions are only defined in the set $|z| < 1$, the intersection

$$v_1^{-1}(0) \cap v_2^{-1}(0)$$

is the set $z = 0$, that is, the curve γ (which is identified via the coordinates with the set $\mathbb{S}^1 \times \{0\}$). Moreover, the behavior of the Neumann eigenfunctions near the set $z = 0$ shows that their intersection is *transverse* in the sense that

$$(2.2) \quad \text{rank}(\nabla v_1(x), \nabla v_2(x)) = 2$$

for all $x \in v_1^{-1}(0) \cap v_2^{-1}(0)$. These properties obviously hold for any other basis $\{v_1, v_2\}$ of the eigenspace corresponding to μ_1 .

Step 2: Convergence of eigenvalues and eigenfunctions. Let us take a one-parameter family of smooth, positive, uniformly bounded functions f_t on M such that $f_t(x) = \epsilon$ in the complement $\Omega^c := M \setminus \Omega$ and $f_t(x) = 1$ if $x \in \Omega$ and the distance between x and $\partial\Omega$, as measured with the metric g_0 , is larger than $1/t$. Here both t and ϵ are positive real numbers, and f_t is assumed to be analytic in t and ϵ . Notice that the smooth functions f_t converge pointwise to the discontinuous function

$$f(x) := \begin{cases} \epsilon & \text{if } x \in \Omega^c, \\ 1 & \text{if } x \in \Omega, \end{cases}$$

as $t \rightarrow \infty$.

Let us now define the one-parameter family of smooth metrics $g_t := f_t g_0$. In the following proposition we consider the behavior of the eigenvalues $\lambda_{k,t}$ and eigenfunctions $u_{k,t}$ of the metric g_t as the parameter ϵ tends to zero and t tends to infinity. Specifically, we are interested in comparing these quantities with the Neumann eigenvalues μ_k and eigenfunctions v_k of the domain Ω , which were introduced in Step 1:

Proposition 2.1. *There is a basis of the eigenfunctions $u_{k,t}$ of the Laplacian on M with the metric g_t and a basis v_k of the Neumann eigenfunctions of the domain Ω such that*

$$\begin{aligned}\lambda_{k,t} - \mu_k &= o(1), \\ \|u_{k,t} - v_k\|_{H^1(\Omega)} &= o(1).\end{aligned}$$

Here we are denoting by $o(1)$ quantities that tend to zero (not uniformly in k) as $\epsilon \searrow 0$ and $t \rightarrow \infty$.

The proof of this result is presented in Section 3, and uses ideas introduced by Colin de Verdière [4].

Step 3: Collapse of eigenvalues. From Proposition 2.1 we infer that the eigenvalues $\lambda_{1,t}$ and $\lambda_{2,t}$ are very close to the first Neumann eigenvalue $\mu_{\mathbb{D}^2}$ provided that t is large enough and ϵ is sufficiently small. Uhlenbeck's theorem [19] implies that, generically, $\lambda_{1,t} \neq \lambda_{2,t}$, so generally if we want to get a sequence of eigenvalues of multiplicity 2 converging to $\mu_{\mathbb{D}^2}$, we need to deform the family of metrics g_t . In the following proposition we show that such a deformation exists. The statement of the proposition is in terms of an ϵ -dependent smooth metric \bar{g}_ϵ , $\epsilon > 0$, whose eigenvalues will be denoted by $\bar{\lambda}_{k,\epsilon}$.

Proposition 2.2. *There exists a positive ϵ -dependent function h_ϵ on M of class C^∞ , a basis of the eigenfunctions $\bar{u}_{k,\epsilon}$ of the Laplacian on M with the metric $\bar{g}_\epsilon := h_\epsilon g_T$, where $T := \epsilon^{-1}$, and a basis v_k of the Neumann eigenfunctions of the domain Ω (with the metric g_0) such that:*

- (i) $\bar{\lambda}_{1,\epsilon}$ has multiplicity 2 and $\bar{\lambda}_{1,\epsilon} = \bar{\lambda}_{2,\epsilon} = \mu_{\mathbb{D}^2}$,
- (ii) $\lim_{\epsilon \searrow 0} \|\bar{u}_{1,\epsilon} - v_1\|_{H^1(\Omega)} = \lim_{\epsilon \searrow 0} \|\bar{u}_{2,\epsilon} - v_2\|_{H^1(\Omega)} = 0$.

Moreover, $\lim_{\epsilon \searrow 0} \|h_\epsilon - 1\|_{C^l(M)} = 0$ for any integer l .

The proof of this result is in Section 4 and exploits the fact that when $\epsilon \searrow 0$ the eigenvalues $\lambda_{1,T}$ and $\lambda_{2,T}$ are very close according to Proposition 2.1 (recall that $\mu_1 = \mu_2 = \mu_{\mathbb{D}^2}$). Then a Hadamard-type formula for the variation of eigenvalues with respect to metric perturbations allows us to use an implicit function theorem argument to construct the function h_ϵ .

Step 4: Transverse intersection of nodal sets. Proposition 2.2 ensures that, for any fixed $\delta > 0$, one can then choose a small ϵ so that the eigenvalue $\bar{\lambda}_{1,\epsilon}$ of the Laplacian computed with the metric \bar{g}_ϵ has multiplicity 2, and that there is a basis $\{\bar{u}_{1,\epsilon}, \bar{u}_{2,\epsilon}\}$ of the corresponding eigenspace such that

$$(2.3) \quad \|\bar{u}_{1,\epsilon} - v_1\|_{H^1(\Omega)} < \delta, \quad \|\bar{u}_{2,\epsilon} - v_2\|_{H^1(\Omega)} < \delta.$$

Here $\{v_1, v_2\}$ is a basis of Neumann eigenfunctions of the domain Ω .

Since the conformal factor h_ϵ defining the metric \bar{g}_ϵ in Proposition 2.2 is arbitrarily close to 1 as $\epsilon \searrow 0$, the definition of the metric g_t implies that

$$\|\bar{g}_\epsilon - g_0\|_{C^l(S)} < C_l \delta$$

for any integer l and any compact subset $S \subset \Omega$, where the constant C_l depends on l and S . Therefore, standard elliptic estimates allow us to promote the H^1

bound (2.3) to a C^1 bound:

$$(2.4) \quad \|\bar{u}_{1,\epsilon} - v_1\|_{C^1(S)} < C\delta, \quad \|\bar{u}_{2,\epsilon} - v_2\|_{C^1(S)} < C\delta.$$

For small enough δ (and hence small ϵ), the transversality property (2.2) allows us to apply Thom's isotopy theorem [1, Section 20.2] to conclude that the nodal set $\bar{u}_{1,\epsilon}^{-1}(0) \cap \bar{u}_{2,\epsilon}^{-1}(0)$ has a connected component in Ω diffeomorphic to

$$\gamma = v_1^{-1}(0) \cap v_2^{-1}(0),$$

and that the corresponding diffeomorphism $\Phi_\epsilon : M \rightarrow M$ can be chosen arbitrarily close to the identity in $C^1(M)$ and different from the identity only in a small neighborhood of γ .

Finally, let us consider the pulled-back metric

$$g := \Phi_\epsilon^* \bar{g}_\epsilon$$

and call

$$u_1 := \bar{u}_{1,\epsilon} \circ \Phi_\epsilon, \quad u_2 := \bar{u}_{2,\epsilon} \circ \Phi_\epsilon$$

the associated eigenfunctions, whose corresponding eigenvalue $\lambda_1 := \bar{\lambda}_{1,\epsilon}$ has multiplicity 2. Accordingly, the nodal set $u_1^{-1}(0) \cap u_2^{-1}(0)$ has a connected component given by the curve γ , which is structurally stable because Eq. (2.2) and the C^1 estimates (2.4) imply

$$\text{rank}(\nabla u_1(x), \nabla u_2(x)) = 2$$

for all $x \in \gamma$. The theorem then follows.

Remark 2.3. The transversality property (2.2) of the intersection of the nodal sets $v_1^{-1}(0)$ and $v_2^{-1}(0)$ is key in the proof, as in particular it allows us to use Thom's isotopy theorem in Step 4.

3. PROOF OF PROPOSITION 2.1

Since the family of smooth functions f_t converges pointwise to the discontinuous function f as $t \rightarrow \infty$, it is convenient to work first with the discontinuous metric

$$g_\epsilon := f g_0 = \begin{cases} \epsilon g_0 & \text{if } x \in \Omega^c, \\ g_0 & \text{if } x \in \Omega, \end{cases}$$

and later we will prove the desired result for the metric g_t provided that t is large enough.

To define the spectrum of the Laplacian associated to the metric g_ϵ , we use the quadratic form

$$Q_\epsilon(\varphi) := \int_M |d\varphi|_\epsilon^2 dV_\epsilon = \int_\Omega |d\varphi|^2 + \epsilon^{\frac{1}{2}} \int_{\Omega^c} |d\varphi|^2,$$

and the natural L^2 norm corresponding to the metric g_ϵ :

$$\|\varphi\|_\epsilon^2 := \int_M \varphi^2 dV_\epsilon = \int_\Omega \varphi^2 + \epsilon^{\frac{3}{2}} \int_{\Omega^c} \varphi^2.$$

Here the subscripts ϵ refer to quantities computed with respect to the metric g_ϵ and we are omitting the subscripts (and indeed the measure in the integrals) when the quantities correspond to the reference metric g_0 . As is well known, the domain of the quadratic form Q_ϵ can be taken to be the Sobolev space $H^1(M)$. (Recall that,

M being compact, this Sobolev space is independent of the smooth metric one uses to define it).

By the min-max principle, the k^{th} eigenvalue $\lambda_{k,\epsilon}$ of this quadratic form is

$$(3.1) \quad \lambda_{k,\epsilon} = \inf_{W \in \mathcal{W}_k} \max_{\varphi \in W \setminus \{0\}} q_\epsilon(\varphi),$$

where \mathcal{W}_k stands for the set of $(k+1)$ -dimensional linear subspaces of $H^1(M)$ and

$$q_\epsilon(\varphi) := \frac{Q_\epsilon(\varphi)}{\|\varphi\|_\epsilon^2}$$

is the Rayleigh quotient. The k^{th} eigenfunction $u_{k,\epsilon}$ is then a minimizer of the above variational problem for $\lambda_{k,\epsilon}$, in the sense that any subspace that minimizes the variational problem can be written as $\text{span}\{u_{0,\epsilon}, \dots, u_{k,\epsilon}\}$.

In what follows we fix an integer k_0 . Let us first prove that the eigenvalues $\lambda_{k,\epsilon}$ are almost upper bounded by μ_k for $k \leq k_0$ in the sense that

$$(3.2) \quad \lambda_{k,\epsilon} \leq \mu_k + O(\epsilon^{\frac{1}{2}}),$$

where $O(\epsilon^m)$ stands for a quantity that is bounded by $C\epsilon^m$. For this purpose, we consider the harmonic extension to the whole manifold M of the Neumann eigenfunctions v_k of the domain Ω :

$$\psi_k := \begin{cases} \hat{v}_k & \text{in } \Omega^c, \\ v_k & \text{in } \Omega, \end{cases}$$

where \hat{v}_k is the solution to the boundary value problem

$$\Delta \hat{v}_k = 0 \text{ in } \Omega^c, \quad \hat{v}_k|_{\partial\Omega} = v_k|_{\partial\Omega}.$$

Recall that Δ is the Laplacian computed with the reference metric g_0 . Throughout this section we shall normalize the Neumann eigenfunctions v_k so that they define an orthonormal basis of $L^2(\Omega)$. Standard elliptic estimates then give us the bound

$$(3.3) \quad \int_{\Omega^c} |d\hat{v}_k|^2 + \int_{\Omega^c} \hat{v}_k^2 \leq C \|v_k\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \leq C \|v_k\|_{H^1(\Omega)}^2 = C(\mu_k + 1),$$

where C is a constant independent of ϵ and k .

For any linear combination $\varphi, \tilde{\varphi} \in \text{span}\{\psi_0, \dots, \psi_k\}$, with $k \leq k_0$, using Eq. (3.3) we obtain that

$$\begin{aligned} \int_M \varphi \tilde{\varphi} dV_\epsilon &= \int_\Omega \varphi \tilde{\varphi} + \epsilon^{\frac{3}{2}} \int_{\Omega^c} \varphi \tilde{\varphi} \\ &= \int_\Omega \varphi \tilde{\varphi} + O(\epsilon^{\frac{3}{2}})(\mu_{k_0} + 1) \|\varphi\|_{L^2(\Omega)} \|\tilde{\varphi}\|_{L^2(\Omega)}, \\ &= \int_\Omega \varphi \tilde{\varphi} + O(\epsilon^{\frac{3}{2}}) \|\varphi\|_{L^2(\Omega)} \|\tilde{\varphi}\|_{L^2(\Omega)}, \end{aligned}$$

where, to pass to the last line, we have absorbed the constant $\mu_{k_0} + 1$ in the $O(\epsilon^{\frac{3}{2}})$ term. Accordingly, for sufficiently small ϵ , the linear space

$$\text{span}\{\psi_0, \dots, \psi_k\},$$

is a $(k+1)$ -dimensional subspace of $H^1(M)$ provided that $k \leq k_0$. In a similar manner we have

$$\begin{aligned} Q_\epsilon(\varphi) &= \int_{\Omega} |d\varphi|^2 + \epsilon^{\frac{1}{2}} \int_{\Omega^c} |d\varphi|^2 \\ &= \int_{\Omega} |d\varphi|^2 + O(\epsilon^{\frac{1}{2}}) \|\varphi\|_{L^2(\Omega)}^2. \end{aligned}$$

Therefore, for $k \leq k_0$ we can estimate $\lambda_{k,\epsilon}$ as

$$\begin{aligned} \lambda_{k,\epsilon} &\leq \max_{\varphi \in \{\psi_0, \dots, \psi_k\}} \frac{Q_\epsilon(\varphi)}{\|\varphi\|_\epsilon^2} \\ &\leq \max_{\varphi \in \{\psi_0, \dots, \psi_k\}} \frac{\int_{\Omega} |d\varphi|^2 + O(\epsilon^{\frac{1}{2}}) \|\varphi\|_{L^2(\Omega)}^2}{\|\varphi\|_{L^2(\Omega)}^2} \\ &= \max_{\chi \in \{v_0, \dots, v_k\}} \frac{\int_{\Omega} |d\chi|^2}{\|\chi\|_{L^2(\Omega)}^2} + O(\epsilon^{\frac{1}{2}}) \\ &= \mu_k + O(\epsilon^{\frac{1}{2}}), \end{aligned}$$

which proves the upper bound (3.2).

Now we proceed to prove the lower bound

$$(3.4) \quad \lambda_{k,\epsilon} \geq \left(1 + O(\epsilon^{\frac{1}{2}})\right) \mu_k$$

for all $k \leq k_0$. To obtain this bound we split $H^1(M)$ into two closed linear subspaces

$$\begin{aligned} \mathcal{H}_1 &:= \{\varphi \in H^1(M) : \Delta\varphi = 0 \text{ in } \Omega^c\}, \\ \mathcal{H}_2 &:= \{\varphi \in H^1(M) : \varphi|_{\Omega^c} \in H_0^1(\Omega^c), \varphi|_{\Omega} = 0\}, \end{aligned}$$

so that $H^1(M) = \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$\int_M g_0(\nabla\varphi_1, \nabla\varphi_2) = 0,$$

for all $\varphi_j \in \mathcal{H}_j$. We will write φ_j for the component of a function $\varphi \in H^1(M)$ in \mathcal{H}_j .

We first note that by choosing a sufficiently small ϵ , the upper bound (3.2) implies that $\lambda_{k,\epsilon} < \mu_{k_0} + 1$ for all $k \leq k_0$, which allows us to reduce the min-max formula (3.1) for $k \leq k_0$ to

$$(3.5) \quad \lambda_{k,\epsilon} = \inf_{W \in \mathcal{W}'_k} \max_{\varphi \in W \setminus \{0\}} q_\epsilon(\varphi),$$

where \mathcal{W}'_k is the set of $(k+1)$ -dimensional subspaces of $H^1(M)$ such that

$$q_\epsilon(\varphi) < \mu_{k_0} + 1$$

for all nonzero $\varphi \in W \subset \mathcal{W}'_k$.

A simple computation shows that

$$(3.6) \quad \|\varphi_2\|_\epsilon^2 = O(\epsilon) \|\varphi\|_\epsilon^2,$$

for any $\varphi \in W \subset \mathcal{W}'_k$ with $k \leq k_0$. Indeed, this relation follows from the estimate

$$\mu_{k_0} + 1 > q_\epsilon(\varphi) = \frac{\|\varphi_2\|_\epsilon^2}{\|\varphi\|_\epsilon^2} \frac{Q_\epsilon(\varphi)}{\|\varphi_2\|_\epsilon^2} \geq \frac{1}{\epsilon} \frac{\|\varphi_2\|_\epsilon^2}{\|\varphi\|_\epsilon^2} \frac{\int_{\Omega^c} |d\varphi_2|^2}{\int_{\Omega^c} \varphi_2^2} \geq \frac{\lambda_{\Omega^c}}{\epsilon} \frac{\|\varphi_2\|_\epsilon^2}{\|\varphi\|_\epsilon^2},$$

where the ϵ -independent constant λ_{Ω^c} is the first Dirichlet eigenvalue of Ω^c . In particular, Eq. (3.6) implies that

$$(3.7) \quad \|\varphi\|_\epsilon^2 = \left(1 + O(\epsilon^{\frac{1}{2}})\right) \|\varphi_1\|_\epsilon^2.$$

Proceeding as in the proof of Eq. (3.3) and using the bound (3.7), we obtain that for $\varphi \in W \subset \mathcal{W}'_k$ with $k \leq k_0$,

$$\int_{\Omega^c} \varphi_1^2 + \int_{\Omega^c} |d\varphi_1|^2 \leq C \int_{\Omega} \varphi_1^2.$$

Combining these two results we conclude that,

$$\begin{aligned} q_\epsilon(\varphi) &= \left(1 + O(\epsilon^{\frac{1}{2}})\right) \frac{\int_M |d\varphi_1|_\epsilon^2 dV_\epsilon + \int_M |d\varphi_2|_\epsilon^2 dV_\epsilon}{\|\varphi_1\|_\epsilon^2} \\ &\geq \left(1 + O(\epsilon^{\frac{1}{2}})\right) \frac{\int_{\Omega} |d\varphi_1|^2}{\int_{\Omega} \varphi_1^2 + \epsilon^{\frac{3}{2}} \int_{\Omega^c} \varphi_1^2} \\ &= \left(1 + O(\epsilon^{\frac{1}{2}})\right) \frac{\int_{\Omega} |d\varphi_1|^2}{\int_{\Omega} \varphi_1^2}, \end{aligned}$$

for any $\varphi \in W \subset \mathcal{W}'_k$ with $k \leq k_0$. Therefore by the min-max principle (3.5) we have

$$\begin{aligned} \lambda_{k,\epsilon} &\geq \left(1 + O(\epsilon^{\frac{1}{2}})\right) \inf_{W \in \mathcal{W}'_k} \max_{\varphi \in W \setminus \{0\}} \frac{\int_{\Omega} |d\varphi_1|^2}{\int_{\Omega} \varphi_1^2} \\ &\geq \left(1 + O(\epsilon^{\frac{1}{2}})\right) \mu_k, \end{aligned}$$

for all $k \leq k_0$, which establishes the lower bound (3.4).

The bounds (3.2) and (3.4) imply that

$$(3.8) \quad \lim_{\epsilon \searrow 0} \|\lambda_{k,\epsilon} - \mu_k\| = 0$$

for all $k \leq k_0$. Since the metric g_ϵ is equal to g_0 in the domain Ω , and the eigenvalues control the H^1 norm, it is standard that the convergence of eigenvalues (3.8) implies that for $k \leq k_0$, there exists a basis of eigenfunctions $u_{k,\epsilon}$ and a basis of Neumann eigenfunctions v_k such that

$$(3.9) \quad \lim_{\epsilon \searrow 0} \|u_{k,\epsilon} - v_k\|_{H^1(\Omega)} = 0.$$

To finish the proof of the proposition, recall that the family of smooth metrics g_t converges pointwise to g_ϵ . It is then well known (see e.g. [2]) that the eigenvalue $\lambda_{k,t}$ converges to $\lambda_{k,\epsilon}$ and that there is a basis of eigenfunctions $u_{k,t}$ converging in $H^1(M)$ to a basis of eigenfunctions $u_{k,\epsilon}$ as $t \rightarrow \infty$. Combining this result with equations (3.8) and (3.9), the proposition follows.

4. PROOF OF PROPOSITION 2.2

In order to construct the conformal factor h_ϵ , we introduce an auxiliary smooth function h_s depending on two real parameters $s \equiv (s_1, s_2) \in \mathbb{R}^2$ that has the form

$$h_s = e^{2s_1\psi_1 + 2s_2\psi_2},$$

where ψ_1 and ψ_2 are smooth functions with zero normal derivative on $\partial\Omega$ that will be fixed later. We denote by $\lambda_{k,\epsilon,s}$ the eigenvalues of the smooth metric

$$g_{\epsilon,s} := h_s g_T,$$

where $T := \epsilon^{-1}$, and by $u_{k,\epsilon,s}$ an orthonormal basis of the corresponding eigenfunctions. By construction we have that $g_{\epsilon,s}$ is analytic in the parameters $\epsilon > 0$ and $s \in \mathbb{R}^2$, and that $g_{\epsilon,0} = g_T$. We also denote by $\mu_{k,s}$ the Neumann eigenvalues of the Laplacian associated with the smooth metric

$$g_{0,s} := h_s g_0$$

in the domain Ω , and by $v_{k,s}$ an orthonormal basis of the corresponding eigenfunctions. By the Kato–Rellich theorem, since the Laplacian for the metric $g_{\epsilon,s}$ is analytic in the parameters $s \in \mathbb{R}^2$ and $\epsilon > 0$, it is well known that we can take the families of eigenvalues $\lambda_{k,\epsilon,s}$ and eigenfunctions $u_{k,\epsilon,s}$ to vary analytically in s and ϵ (this result is independent of the multiplicities of the eigenvalues). The same theorem also implies that the Neumann eigenvalues $\mu_{k,s}$ and eigenfunctions $v_{k,s}$ are analytic in s , and in particular

$$(4.1) \quad \lim_{s \rightarrow 0} |\mu_{k,s} - \mu_k| = 0, \quad \lim_{s \rightarrow 0} \|v_{k,s} - v_k\|_{H^1(\Omega)} = 0.$$

To construct the desired function h_ϵ we will apply a version of the implicit function theorem to the map $\Lambda : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$\Lambda(\epsilon, s) := \begin{cases} (\lambda_{1,\epsilon,s} - \mu_{\mathbb{D}^2}, \lambda_{2,\epsilon,s} - \mu_{\mathbb{D}^2}) & \text{if } \epsilon > 0, \\ (\mu_{1,s} - \mu_{\mathbb{D}^2}, \mu_{2,s} - \mu_{\mathbb{D}^2}) & \text{if } \epsilon = 0. \end{cases}$$

From the analytic dependence of the eigenvalues with ϵ and s , we have that $\Lambda(\epsilon, s)$ depends analytically on the variable $\epsilon \in (0, \infty)$ for any fixed $s \in \mathbb{R}^2$ and also on the variable $s \in \mathbb{R}^2$ for any fixed $\epsilon \in [0, \infty)$. In particular, Λ is differentiable in its second variable when the first is fixed. Moreover, Λ is continuous in both variables at the point $(0, s_0)$ for any $s_0 \in \mathbb{R}^2$ and $\Lambda(0, 0) = (0, 0)$. This is proved by proceeding exactly as in the proof of Proposition 2.1, cf. Section 3, to show that

$$\lim_{\epsilon \searrow 0} |\lambda_{k,\epsilon,s} - \mu_{k,s}| = 0,$$

and

$$(4.2) \quad \lim_{\epsilon \searrow 0} \|u_{k,\epsilon,s} - v_{k,s}\|_{H^1(\Omega)} = 0,$$

for any fixed $s \in \mathbb{R}^2$ (in the proof one only has to change the reference metric g_0 by the metric $g_{0,s}$). Since this property does not depend on the particular way that the point (ϵ, s) tends to $(0, s_0)$, we conclude that the map Λ is continuous on its domain.

With the aim of using an implicit function theorem, we now need to evaluate the derivative of Λ with respect to s at the point $(0, 0)$. To do this we first calculate the derivative of the Laplacian Δ_s , computed with the metric $g_{0,s}$, at the point $s = 0$, acting on the Neumann eigenfunction v_k . Noticing that $\Delta_s v_k$ reads as

$$\Delta_s v_k = \frac{1}{2} h_s^{-2} g_0(\nabla h_s, \nabla v_k) - \mu_k h_s^{-1} v_k,$$

where ∇ is computed with the reference metric g_0 , a straightforward computation taking derivatives of this expression with respect to the parameter s_i , $i \in \{1, 2\}$, and using the form of the conformal factor h_s , shows that

$$(4.3) \quad D_{s_i} \Delta_s(0) v_k = g_0(\nabla \psi_i, \nabla v_k) + 2\mu_k \psi_i v_k.$$

Now we can take derivatives of the eigenvalue equation $\Delta_s v_{k,s} = -\mu_{k,s} v_{k,s}$ with respect to s_i (recall that the families of eigenfunctions $v_{k,s}$ and $\mu_{k,s}$ were chosen to be analytic with respect to s), and evaluate at $s = 0$. If we multiply the resulting expression by v_k and integrate over the domain Ω (with respect to the measure induced by g_0 , which we omit for the sake of simplicity) we easily get

$$(4.4) \quad D_{s_i} \mu_{k,s}(0) = - \int_{\Omega} v_k D_{s_i} \Delta_s(0) v_k - \int_{\Omega} v_k \Delta(D_{s_i} v_{k,s}(0)) - \mu_k \int_{\Omega} v_k D_{s_i} v_{k,s}(0).$$

To obtain this formula we have used that $\int_{\Omega} v_k^2 = 1$.

Noticing that the unit vector field ν_s orthogonal to the boundary $\partial\Omega$ with respect to the metric $g_{0,s}$ is given by

$$\nu_s = h_s^{-\frac{1}{2}} \nu,$$

with ν the unit normal field of $\partial\Omega$ computed with g_0 , we infer that the Neumann eigenfunction $v_{k,s}$ also satisfies the Neumann boundary condition with respect to the normal field ν , that is, $\partial_{\nu} v_{k,s} = 0$. In particular, taking derivatives with respect to s_i in this expression we get $\partial_{\nu}(D_{s_i} v_{k,s}(0)) = 0$.

Accordingly, the $D_{s_i} v_{k,s}(0)$ terms in Eq. (4.4) cancel after integration by parts, using that $\Delta v_k = -\mu_k v_k$. Finally, putting together Eqs. (4.3) and (4.4) we obtain a Hadamard-type formula for the variation of the Neumann eigenvalue $\mu_{k,s}$:

$$\begin{aligned} D_{s_i} \mu_{k,s}(0) &= - \int_{\Omega} v_k g_0(\nabla \psi_i, \nabla v_k) - 2\mu_k \int_{\Omega} \psi_i v_k^2 \\ &= \frac{1}{2} \int_{\Omega} v_k^2 (\Delta \psi_i - 4\mu_k \psi_i). \end{aligned}$$

Here we have integrated by parts and used that $\partial_{\nu} \psi_i = 0$, by assumption, in order to pass to the second equality.

This equation and the fact that $\mu_1 = \mu_2 = \mu_{\mathbb{D}^2}$ imply that the expression for the derivative $D_s \Lambda(0, 0)$ is:

$$D_s \Lambda(0, 0) = \frac{1}{2} \begin{pmatrix} \int_{\Omega} v_1^2 (\Delta \psi_1 - 4\mu_{\mathbb{D}^2} \psi_1) & \int_{\Omega} v_1^2 (\Delta \psi_2 - 4\mu_{\mathbb{D}^2} \psi_2) \\ \int_{\Omega} v_2^2 (\Delta \psi_1 - 4\mu_{\mathbb{D}^2} \psi_1) & \int_{\Omega} v_2^2 (\Delta \psi_2 - 4\mu_{\mathbb{D}^2} \psi_2) \end{pmatrix}.$$

In order to apply an implicit function theorem to the map Λ , in the following lemma we compute the rank of the matrix $D_s \Lambda(0, 0)$ for a suitable choice of the functions ψ_1, ψ_2 . The lemma is stated in terms of a function \hat{v}_{k_0} that is any smooth extension to the whole manifold M of the Neumann eigenfunction v_{k_0} of the domain Ω .

Lemma 4.1. *There exists a positive integer k_0 such that*

$$\text{rank} [D_s \Lambda(0, 0)] = 2$$

with the choice $\psi_1 = 1$ and $\psi_2 = \hat{v}_{k_0}$.

Proof. First notice that, with this choice, the functions ψ_1 and ψ_2 satisfy the Neumann condition $\partial_\nu \psi_i = 0$ on $\partial\Omega$. Since $\int_\Omega v_k^2 = 1$ by assumption (we are considering an orthonormal basis), the derivative $D_s \Lambda(0, 0)$ reads as

$$D_s \Lambda(0, 0) = -\frac{1}{2} \begin{pmatrix} 4\mu_{\mathbb{D}^2} & (\mu_{k_0} + 4\mu_{\mathbb{D}^2}) \int_\Omega v_1^2 v_{k_0} \\ 4\mu_{\mathbb{D}^2} & (\mu_{k_0} + 4\mu_{\mathbb{D}^2}) \int_\Omega v_2^2 v_{k_0} \end{pmatrix}.$$

The problem then reduces to finding a Neumann eigenfunction v_{k_0} such that

$$\int_\Omega (v_1^2 - v_2^2) v_{k_0} \neq 0.$$

It is obvious that such eigenfunction exists because the set of Neumann eigenfunctions $\{v_k\}_{k=0}^\infty$ spans $L^2(\Omega)$ and the quantity $v_1^2 - v_2^2$ is not identically zero due to the fact that v_1 and v_2 are not proportional. This proves the lemma. \square

We can now use the implicit function theorem [12, Theorem B] to obtain a constant $\epsilon_0 > 0$, an open neighborhood U of $0 \in \mathbb{R}^2$ and a function $R : [0, \epsilon_0) \rightarrow U$ such that:

- $R(0) = 0$,
- $\Lambda(\epsilon, R(\epsilon)) = \Lambda(0, 0) = 0$ for all $\epsilon \in [0, \epsilon_0)$, and
- R is continuous at $\epsilon = 0$, so in particular $\lim_{\epsilon \searrow 0} R(\epsilon) = 0$.

We can define the desired conformal factor h_ϵ as

$$h_\epsilon := h_{R(\epsilon)},$$

so that the continuity of R implies that $\lim_{\epsilon \searrow 0} \|h_\epsilon - 1\|_{C^l(M)} = 0$ for any integer l . With this definition of h_ϵ we have that the metric \bar{g}_ϵ and the eigenvalues $\bar{\lambda}_{k,\epsilon}$ in the statement of the proposition are given by

$$\bar{g}_\epsilon = g_{\epsilon, R(\epsilon)}, \quad \bar{\lambda}_{k,\epsilon} = \lambda_{k,\epsilon, R(\epsilon)}.$$

Accordingly, since $\Lambda(\epsilon, R(\epsilon)) = 0$ for $\epsilon \in [0, \epsilon_0)$, we conclude that

$$\bar{\lambda}_{1,\epsilon} = \bar{\lambda}_{2,\epsilon} = \mu_{\mathbb{D}^2}$$

provided that $\epsilon > 0$ is small enough. The multiplicity of this eigenvalue is exactly 2 because $\lim_{\epsilon \searrow 0} |\bar{\lambda}_{3,\epsilon} - \mu_3| = 0$ and $\mu_3 > \mu_{\mathbb{D}^2}$.

Concerning the eigenfunctions $\bar{u}_{k,\epsilon}$ associated to the eigenvalues $\bar{\lambda}_{k,\epsilon}$, we observe that

$$\bar{u}_{k,\epsilon} = u_{k,\epsilon, R(\epsilon)},$$

and therefore Eqs. (4.1) and (4.2) imply

$$\lim_{\epsilon \searrow 0} \|\bar{u}_{1,\epsilon} - v_1\|_{H^1(\Omega)} = \lim_{\epsilon \searrow 0} \|\bar{u}_{2,\epsilon} - v_2\|_{H^1(\Omega)} = 0,$$

thus completing the proof of the proposition.

5. HIGHER DIMENSIONAL ANALOGS OF THE MAIN THEOREM

In Theorem 1.1 we have shown the existence of a Riemannian metric for any closed 3-manifold whose first nontrivial eigenvalue has multiplicity 2 (i.e., $\lambda_1 = \lambda_2 \neq \lambda_3$) and a prescribed component of codimension 2 (that is, a knot) in the nodal set $u^{-1}(0)$ of the complex-valued eigenfunction $u = u_1 + iu_2$. In this section we shall show that essentially the same argument enables us to construct a metric on

any closed d -dimensional manifold, whose first nontrivial eigenvalue has multiplicity $m \geq 2$ and a prescribed component of codimension m in the set

$$u_1^{-1}(0) \cap \cdots \cap u_m^{-1}(0).$$

However, a technical condition makes the statement considerably more involved in the general case. This condition is associated with the crucial requirement that the nodal set be structurally stable. In the situation covered by the main theorem, the structural stability follows from the important equation (2.2), which plays a crucial role in the proof (see Remark 2.3). The higher dimensional analog of that relation is the requirement that the intersection of the nodal sets $u_1^{-1}(0), \dots, u_m^{-1}(0)$ is transverse on $\Sigma \subset \cap_{k=1}^m u_k^{-1}(0)$, that is,

$$(5.1) \quad \text{rank}(\nabla u_1(x), \dots, \nabla u_m(x)) = m$$

for all $x \in \Sigma$.

We shall next state the higher dimensional analog of Theorem 1.1. For this, let Σ be a compact embedded submanifold of a d -dimensional manifold M . Throughout we will assume that Σ has codimension $m \geq 2$ and is connected and boundaryless. Our result shows that Σ is a connected component of the transverse intersection of m independent eigenfunctions corresponding to the same eigenvalue λ_1 of the Laplacian on M for some metric. For the transversality condition (5.1) to hold, a topological obstruction on Σ is that its normal bundle must be trivial. Geometrically, this is equivalent to the assertion that a small tubular neighborhood of the submanifold Σ must be diffeomorphic to $\Sigma \times \mathbb{D}^m$, where \mathbb{D}^m denotes the unit m -dimensional disk. Since a knot always has trivial normal bundle [18], Theorem 1.1 corresponds exactly to the case $d = 3$ and $m = 2$. The case $m = 1$, i.e. codimension one hypersurfaces embedded in M , was covered in [10].

Theorem 5.1. *Let Σ be a codimension m compact submanifold of M with trivial normal bundle, $m \geq 2$. Then there exists a Riemannian metric g on M such that its first nontrivial eigenvalue has multiplicity m (i.e., $\lambda_1 = \cdots = \lambda_m \neq \lambda_{m+1}$) and Σ is a connected component of the transverse intersection of the nodal sets $u_1^{-1}(0) \cap \cdots \cap u_m^{-1}(0)$. Here $\{u_1, \dots, u_m\}$ is any basis of the eigenspace corresponding to λ_1 .*

The proof of this result goes almost exactly as the proof of Theorem 1.1. First, we construct a reference metric g_0 on M such that

$$g_0|_{\Omega} = g_{\Sigma} + |dz|^2,$$

where Ω is a tubular neighborhood of Σ , which can be identified with $\Sigma \times \mathbb{D}^m$ because Σ has trivial normal bundle, and $z \in \mathbb{D}^m$ are coordinates parametrizing the disk \mathbb{D}^m . We take the metric g_{Σ} on Σ such that the first nontrivial eigenvalue of the Laplacian on Σ with g_{Σ} is bigger than the first Neumann eigenvalue $\mu_{\mathbb{D}^m}$ of \mathbb{D}^m . Therefore, the Neumann Laplacian in Ω with the metric g_0 has eigenvalues $\mu_0 = 0$,

$$\mu_1 = \cdots = \mu_m = \mu_{\mathbb{D}^m},$$

and $\mu_k > \mu_{\mathbb{D}^m}$ for $k \geq m+1$. The eigenspace of μ_1 is then m -dimensional, with a particular orthogonal basis being

$$v_k = r^{1-\frac{m}{2}} J_{\frac{m}{2}}(\sqrt{\mu_{\mathbb{D}^m}} r) Y_{1,k}(\theta) = C_m z_k + O(|z|^2),$$

where J_i denotes the i^{th} order Bessel function and $Y_{1,k}$, $1 \leq k \leq m$, are the first order spherical harmonics on \mathbb{S}^{m-1} . The explicit expression of the constant C_m is not relevant for our purposes. From this formula for the eigenfunctions it is easy to check that the intersection

$$v_1^{-1}(0) \cap \cdots \cap v_m^{-1}(0)$$

is transverse and coincides with the submanifold Σ . The same holds for any other basis of the eigenspace corresponding to μ_1 .

Step 2 is the same as in the proof of Theorem 1.1, so that we can construct a 1-parameter family of smooth metrics $g_t = f_t g_0$ such that the corresponding eigenvalues $\lambda_{k,t}$ and eigenfunctions $v_{k,t}$ behave when $t \rightarrow \infty$ as in Proposition 2.1. In Step 3 we can prove a result analogous to Proposition 2.2. The only difficulty is in choosing the functions ψ_i (see Section 4) for the conformal factor

$$h_s = e^{2s_1\psi_1 + \cdots + 2s_m\psi_m},$$

where $s \equiv (s_1, \dots, s_m) \in \mathbb{R}^m$. The conditions that these functions must satisfy are that their normal derivatives on $\partial\Omega$ must be zero, and the differential $D_s\Lambda(0,0)$ has maximal rank, where the map $\Lambda : \mathbb{R}^+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as:

$$\Lambda(\epsilon, s) := \begin{cases} (\lambda_{1,\epsilon,s} - \mu_{\mathbb{D}^m}, \dots, \lambda_{m,\epsilon,s} - \mu_{\mathbb{D}^m}) & \text{if } \epsilon > 0, \\ (\mu_{1,s} - \mu_{\mathbb{D}^m}, \dots, \mu_{m,s} - \mu_{\mathbb{D}^m}) & \text{if } \epsilon = 0. \end{cases}$$

The quantities $\lambda_{k,\epsilon,s}$ and $\mu_{k,s}$ are the eigenvalues corresponding to the metrics $g_{\epsilon,s} := h_s g_T$, $T = \epsilon^{-1}$, and $g_{0,s} := h_s g_0$, respectively, with their associated orthonormal eigenfunctions $u_{k,\epsilon,s}$ and $v_{k,s}$. By Kato–Rellich we can assume that all these quantities are analytic in the variables $\epsilon > 0$ and $s \in \mathbb{R}^m$.

The derivative $D_s\Lambda(0,0)$ can be computed using a Hadamard-type formula as in Section 4, thus yielding:

$$(D_s\Lambda(0,0))_{ij} = \frac{1}{2} \int_{\Omega} v_i^2 ((d-2)\Delta\psi_j - 4\mu_{\mathbb{D}^m}\psi_j)$$

for $1 \leq i, j \leq m$. To compute the rank of this matrix, we can prove an equivalent lemma to Lemma 4.1 by noting that the transversality of the nodal sets of v_k , $1 \leq k \leq m$, implies that the functions v_k^2 are linearly independent and hence their span V is m -dimensional. This allows us to choose m linearly independent Neumann eigenfunctions $\{v_{k_1}, \dots, v_{k_m}\}$ so that no function in their span is orthogonal to the subspace V (these eigenfunctions exist because the set of Neumann eigenfunctions is a basis of $L^2(\Omega)$).

Therefore, if we set

$$\psi_i = \hat{v}_{k_i}$$

for $i \in \{1, \dots, m\}$, where \hat{v}_k denotes any smooth extension to the whole manifold M of the Neumann eigenfunction v_k , it follows that the vectors

$$w_i := -\frac{1}{2} \left(\int_{\Omega} ((d-2)\mu_{k_1} + 4\mu_{\mathbb{D}^m}) v_i^2 v_{k_1}, \dots, \int_{\Omega} ((d-2)\mu_{k_m} + 4\mu_{\mathbb{D}^m}) v_i^2 v_{k_m} \right)$$

for $1 \leq i \leq m$, are linearly independent, where each vector w_i is the i^{th} row of the matrix $D_s\Lambda(0,0)$. Indeed, if w_i were linearly dependent, there would exist a

nontrivial set of constants $\{c_i\}_{i=1}^m \subset \mathbb{R}$ such that

$$\sum_{i=1}^m c_i \int_{\Omega} v_i^2 v_{k_j} = 0$$

for all $1 \leq j \leq m$. Hence, the function $F := \sum_{i=1}^m c_i v_i^2$ would be a linear combination of the functions v_i^2 that is orthogonal to each v_{k_j} . By the construction of the Neumann eigenfunctions v_{k_j} , this would imply that $F = 0$ and from the linear independence of the set $\{v_i^2\}_{i=1}^m$ we would obtain that $c_i = 0$ for every $1 \leq i \leq m$, thus proving the claim. Notice that, as in the proof of Proposition 2.2, we can take $\psi_1 = 1$.

The rest of the proof goes exactly as in Section 4 using the implicit function theorem [12, Theorem B] to prove a result analogous to Proposition 2.2, and Thom's isotopy theorem in Step 4, so the details are omitted.

Remark 5.2. Theorem 5.1 has the following striking corollary: since any exotic sphere of dimension m embeds smoothly in \mathbb{S}^{2m} with trivial normal bundle [14], we conclude that there is a metric on \mathbb{S}^{2m} whose first nontrivial eigenvalue has multiplicity m and the exotic sphere is a component of the transverse intersection of the nodal sets $u_1^{-1}(0) \cap \cdots \cap u_m^{-1}(0)$. Recall that an exotic sphere is a smooth manifold that is homeomorphic but not diffeomorphic to the standard sphere.

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